(X, g^{TX}) even-dimensional, oriented $m = \dim X$

Spin

$$\mathrm{Spin}(X) \xrightarrow{\mathbb{Z}_2} SO(X)$$

$$\mathrm{Spin}(X) \xrightarrow{\mathrm{Spin}(m)} X \xleftarrow{SO(m)} SO(X)$$

$$S^{TX\pm} := \mathrm{Spin}(X) \times_{\mathrm{Spin}(m)} S^{\pm}, \quad h^{S^{TX}} \text{ induced by } h^S$$

Lemma: If locally $\nabla^{TX} = d + \Gamma^{TX}$, with $\Gamma^{TX} \in \Omega^1(U, so(m))$

Then: $$\nabla^{S^{TX}} := d + \frac{1}{4} \sum_{i,j=1}^{m} \langle \Gamma^{TX}(\cdot) e_i, e_j \rangle_{g^{TX}} c(e_i) c(e_j)$$

defines a Hermitian Clifford connection on S^{TX}

Pf: Recall that $\rho : \mathrm{Spin}(m) \longrightarrow SO(m)$

$$\rho : spin(m) \xrightarrow{\sim} so(m) \ni A$$

HW 5.2

$$\rho^{-1}(A) = \frac{1}{4} \sum \langle A e_i, e_j \rangle c(e_i) c(e_j)$$

connection form on principal bundle

$\nabla^{S^{TX}}$ is globally well-defined.

$\nabla^{S^{TX}}$ preserves $S^{TX\pm}$ and $h^{S^{TX}}$

We need to verify $\forall\ a \in TX$

$$[\nabla^{S^{TX}}, c(a)] = c(\nabla^{TX} a) \quad \text{on } S^{TX}$$

Locally $\{e_i\}$ ONB of TX

$$a = e_k$$

$$\left[d + \frac{1}{4} \sum_{i,j} \langle \Gamma^{TX} e_i, e_j \rangle c(e_i) c(e_j),\ c(e_k)\right]$$

$$= \frac{1}{4}\sum_{i,j} \langle \Gamma^{TX} e_i, e_j\rangle \underbrace{[c(e_i)c(e_j), c(e_k)]}_{c(e_i)c(e_j)c(e_k) - c(e_k)c(e_i)c(e_j)}$$

$k \notin \{i,j\}$

$k \in \{i,j\}$

$\in so(M)$

$$= \frac{1}{4}\sum_{\substack{j \\ i=k}} \langle \Gamma^{TX} e_k, e_j\rangle (2c(e_j)) + \frac{1}{4}\sum_{\substack{i \\ j=k}} \langle \Gamma^{TX} e_i, e_k\rangle (-2c(e_i))$$

$$= \sum_j \langle \Gamma^{TX} e_k, e_j\rangle c(e_j)$$

$$= c(\Gamma^{TX} e_k) = c(\nabla^{TX} e_k) \qquad \#$$

Prop:

$$R^{S^{TX}} = \frac{1}{4}\sum_{i,j} \langle R^{TX} e_i, e_j\rangle_{g^{TX}} c(e_i)c(e_j)$$

Riemann curvature tensor $R^{TX} = (\nabla^{TX})^2$

Thm: X Spin, (E, h^E, ∇^E) Clifford module on X. Then $\exists$ a vector bundle W on X s.t. $E \simeq S^{TX} \otimes W$

and $\nabla^E = \nabla^{S^{TX}} \otimes Id_W + Id_{S^{TX}} \otimes \nabla^W$

Proof: Define $W_x = \mathrm{Hom}_{C(TX)_x \otimes_{\mathbb{R}} \mathbb{C}}(S^{TX}_x, E_x)$

It is clear $\dim W_x$ is constant on X.

$\xrightarrow[\text{local frame}]{}$ W is a complex vector bundle on X

s.t. $E \cong S^{TX} \otimes W$

Now for $s \in C^\infty(X, S^{TX})$ ③

$\omega \in C^\infty(X, W) = C^\infty(X, \mathrm{Hom}_{C(V)}(S^{TX}, E))$

$(\nabla^W \omega)\otimes s := \nabla^E(\omega \otimes s) - \omega \otimes (\nabla^{S^{TX}} s)$

HW 5.3 : ∇^W is well-defined connection on W. #

When X is not spin, m = even, E Clifford module on X

then locally $S^{TX}|_U$ is well-defined

$E|_U \simeq S^{TX}|_U \otimes W$

W vector bundle on U.

Def/Lemma: $R^{E/S^{TX}} \in \Omega^2(X, \mathrm{End}_{C(V)}(E))$ is defined as

$$R^{E/S^{TX}} := R^E - \frac{1}{4}\sum_{j,k} \langle R^{TX} e_j, e_k\rangle_{g^{TX}} c(e_j)\, c(e_k)$$

$$\underset{\text{locally}}{=} R^W|_U$$

we do not need X to be spin (but m = even).

V.3 Lichnerowicz formula

Def (Bochner Laplacian): (E, h^E) ∇^E Hermitian connection

$$\Delta^E := -\sum_{i=1}^m \left(\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla^{TX}_{e_i} e_i}\right) \curvearrowright C^\infty(X, E)$$

$\{e_i\}$ ONB of (TX, g^{TX})

For $s_1, s_2 \in C^\infty(X, E)$, set

$$\beta = \langle \nabla^E s_1, s_2\rangle_{h^E} \in \Omega^1(X)$$

$$\mathrm{Tr}[\nabla^{T^*X}\beta] = \sum e_i \langle \nabla^E_{e_i} s_1, s_2\rangle_{h^E} - \langle \nabla^E_{\nabla^{TX}_{e_i} e_i} s_1, s_2\rangle_{h^E}$$

$$= \sum \langle (\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla^{TX}_{e_i} e_i}) s_1, s_2\rangle_{h^E}$$

$$+ \langle \nabla^E_{e_i} s_1, \nabla^E_{e_i} s_2\rangle_{h^E}$$

$$\Rightarrow \quad \langle \Delta^E s_1, s_2\rangle_{L^2} = \langle \nabla^E s_1, \nabla^E s_2\rangle_{L^2}$$

$\Delta^E = (\nabla^E)^* \nabla^E \geq 0$, symmetric.

where $\nabla^E : C^\infty(X, E) \to \Omega^1(X, E)$

$(\nabla^E)^*$ formal adjoint.

<u>Thm</u> (Lichnerowicz formula) (X, g^{TX}) oriented, $\dim = m =$ even

(E, h^E, ∇^E) Clifford module

∇^E Hermitian Clifford connection

$$D^E := \sum_{j=1}^m c(e_j) \nabla^E_{e_j} \quad \text{Dirac op.}$$

<u>Then</u>

$$(D^E)^2 = \Delta^E + \frac{r^X}{4} + \sum_{i<j} \underbrace{R^{E/S}(e_i, e_j)}_{\mathrm{End}_{C(V)}(E)} c(e_i) c(e_j)$$

where $r^X := \sum_{i,j} \langle R^{TX}(e_i, e_j) e_j, e_i\rangle_{g^{TX}}$

denotes the scalar curvature of (X, g^{TX}).

<u>Proof</u>:

$$(D^E)^2 = \Big(\sum_{j=1}^m c(e_j) \nabla^E_{e_j}\Big)^2$$

$$= \frac{1}{2}\Big(\sum_{i,j} c(e_i)\nabla_{e_i} c(e_j)\nabla_{e_j} + c(e_j)\nabla_{e_j} c(e_i)\nabla_{e_i}\Big)$$

$$= \frac{1}{2}\sum \Big\{ c(e_i)\big([\nabla_{e_i}, c(e_j)] + c(e_j)\nabla_{e_i}\big)\nabla_{e_j} + c(e_j)\big([\nabla_{e_j}, c(e_i)] + c(e_i)\nabla_{e_j}\big)\nabla_{e_i} \Big\}$$

$$= \frac{1}{2}\sum \Big\{ c(e_i)c(e_j)\nabla_{e_i}\nabla_{e_j} + c(e_j)c(e_i)\nabla_{e_j}\nabla_{e_i} + c(e_i)c(\nabla^{TX}_{e_i}e_j)\nabla_{e_j} + c(e_j)c(\nabla^{TX}_{e_j}e_i)\nabla_{e_i} \Big\}$$

$$= \frac{1}{2}\sum \Big(\overbrace{c(e_i)c(e_j) + c(e_j)c(e_i)}^{-2\delta_{ij}}\Big)\nabla_{e_i}\nabla_{e_j}\Big) + \frac{1}{2}\sum c(e_j)c(e_i)[\nabla_{e_j}, \nabla_{e_i}] + \sum \langle \nabla_{e_i}e_j, e_k\rangle c(e_i)c(e_k)\nabla_{e_j}$$

$$= -\sum_j (\nabla_{e_j})^2 + \sum_{i,j,k}\langle \nabla^{TX}_{e_i}e_j, e_k\rangle c(e_i)c(e_k)\nabla_{e_j} + \frac{1}{2}\sum_{i,j} c(e_i)c(e_j)[\nabla_{e_i}, \nabla_{e_j}]$$

$$\langle \nabla^{TX}_{e_i}e_j, e_k\rangle = -\langle e_j, \nabla^{TX}_{e_i}e_k\rangle$$

$$\sum \langle \nabla^{TX}_{e_i}e_j, e_k\rangle c(e_i)c(e_k)\nabla_{e_j}$$

$$= -\sum_{i,k} c(e_i)c(e_k)\nabla^{E}_{\nabla^{TX}_{e_i}e_k}$$

$$= -\sum_i c(e_i)^2\nabla^{E}_{\nabla^{TX}_{e_i}e_i} - \frac{1}{2}\sum_{i\neq k} c(e_i)c(e_k)\big(\nabla^{E}_{\nabla^{TX}_{e_i}e_k} - \nabla^{E}_{\nabla^{TX}_{e_k}e_i}\big)$$

$$= \sum_i \nabla^E_{\nabla^{TX}_{e_i} e_i} - \frac{1}{2} \sum_{i,k} c(e_i) c(e_k) \nabla^E_{[e_i, e_k]}$$ ⑥

Then we get

$$(D^E)^2 = \Delta^E + \frac{1}{2} \sum_{i,j} c(e_i) c(e_j) R^E(e_i, e_j)$$

Recall

$$R^{E/S} = R^E - \frac{1}{4} \sum_{k,l} \langle R^{TX} e_k, e_l \rangle_{g^{TX}} c(e_k) c(e_l)$$

$$\overset{\cap}{\mathrm{End}_{C(V)}(E)} \rightsquigarrow \underline{R^{E/S}(e_i, e_j) c(e_i) c(e_j) = c(e_i) c(e_j) R^{E/S}(e_i, e_j)}$$

Now $\quad (*) = \sum_{i,j,k,l} \langle R^{TX}(e_i, e_j) e_k, e_l \rangle_{g^{TX}} c(e_i) c(e_j) c(e_k) c(e_l)$

<u>Claim</u>: $\quad (*) = 2 r^X$

Set $\quad R_{ijkl} = \langle R^{TX}(e_j, e_i) e_k, e_l \rangle_{g^{TX}}$

<u>Properties</u>: $\begin{cases} R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk} \\ R_{ijkl} + R_{jkil} + R_{kijl} = 0 \end{cases}$

Recall $\quad r^X = \sum_{i,j} R_{ijij}$

$$c_j = c(e_j)$$

$$(*) = \sum R_{jikl} c_i c_j c_k c_l$$

$$= \sum_{\substack{i \neq k \\ j \neq k}} R_{jikl} c_i c_j c_k c_l + \sum_{\substack{i = k \\ j \neq k}} + \sum_{\substack{i \neq k \\ j = k}}$$

$$\frac{1}{3}\sum_{\substack{i\neq k\\ j\neq k\\ i\neq j}}\left(R_{jikl}\,\underline{c_i c_j}\,c_k + R_{ikjl}\,\underline{c_k c_i}\,c_j + R_{kjil}\,\underline{c_j c_k c_i}\right)c_l$$

$$= \frac{1}{3}\sum_{\substack{i\neq k\\ j\neq k\\ i\neq j}}\left(R_{jikl}+R_{ikjl}+R_{kjil}\right)c_i c_j c_k c_l = 0$$

$$(*) = \sum_{\substack{i=k\\ j\neq k}} R_{jiil}\,\underbrace{c_i c_j c_i c_l}_{c_j c_l} + \sum_{\substack{j=k\\ i\neq k}} R_{jijl}\,\underbrace{c_i c_j c_j c_l}_{-c_i c_l}$$

$$= -2\sum_{i,j,l} R_{jijl}\,c_i c_l$$

$$= \underset{\substack{\uparrow\\ i=l}}{2r^x} - \underbrace{\sum_{\substack{i\neq l\\ j}} R_{jijl}\,(\underbrace{c_i c_l + c_l c_i}_{\delta_{il}})}_{=0}$$

$c_i^2=-1$

$$= 2r^x$$

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